Imperial College
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Topic 7

Linearity, Superposition & Thevenin Equivalent Circuits

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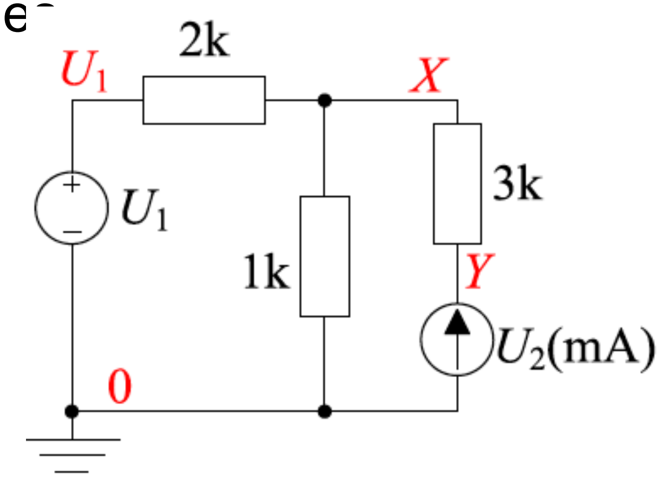
Linearity Theorem

- Suppose we use variables instead of fixed values for all of the *fixed (or independent)* voltage and current sources. We can then use nodal analysis to find all node voltages in terms of the source values



  ① Label all the nodes

  ② KCL equations: $\frac{X-U_1}{2} + \frac{X}{1} + \frac{X-Y}{3} = 0$

  $$\frac{Y-X}{3} + (-U_2) = 0$$

  ③ Solve for the node voltages

  $$X = \tfrac{1}{3}U_1 + \tfrac{2}{3}U_2, \quad Y = \tfrac{1}{3}U_1 + \tfrac{11}{3}U_2$$

- Steps (2) and (3) never involve multiplying two source values together, so:

Linearity Theorem: For any circuit containing resistors and independent voltage and current sources, every node voltage and branch current is a linear function of the source values and has the form $\sum a_i U_i$ where the U_i are the source values and the a_i are suitably dimensioned constants.

Implications of Linearity

- A linear circuit can therefore be described as:

  Effect = **Linear Function F** (Causes) or

  V_{effect} = a_1 x Cause_1 + a_2 x Cause_2 + ……. (a_1, a_2 ... are constants)

- There are TWO important properties in a linear circuits:

  1. **Proportionality** – If you multiply a **cause** by a factor M, the **effect** is also multiplied by the same factor M.
  2. **Superposition** – You can find the effects produce by two causes SEPARATELY, and COMBINE (i.e. add) them together to find the effect of both causes. In other words:

     If Effect_1 = F (Cause_1)

     Effect_2 = F (Cause_2), where F is a **linear function** (i.e. linear circuit)

     then Total Effect = F (Cause_1 + Cause_2) = Effect_1 + Effect_2

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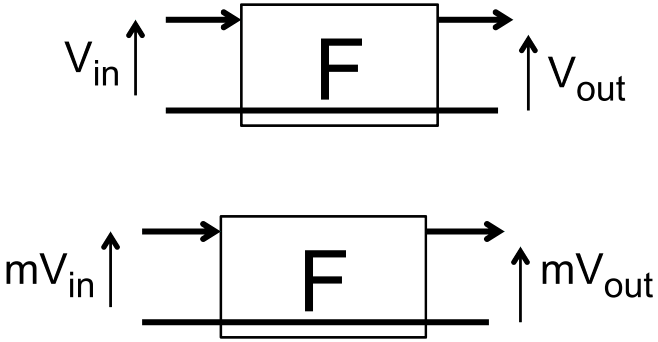
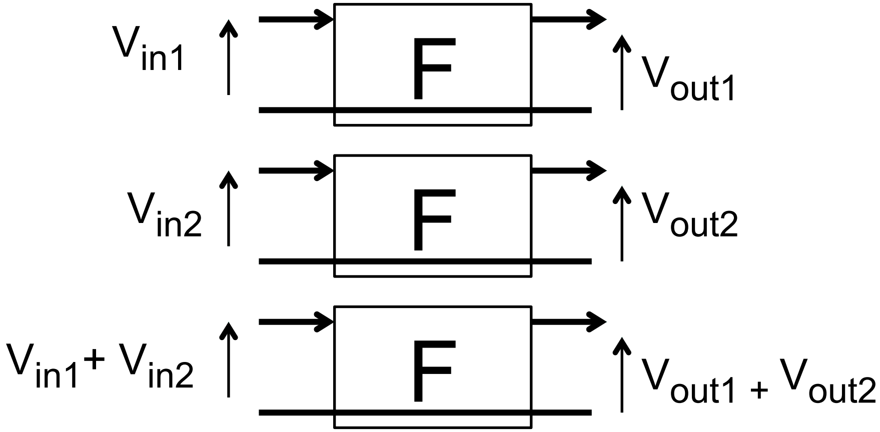
Implications of Linearity

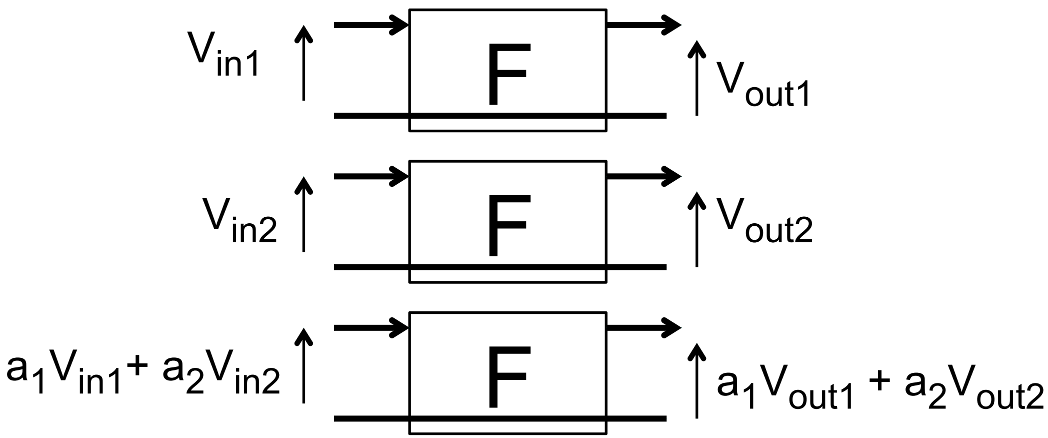
Proportionality



Superposition

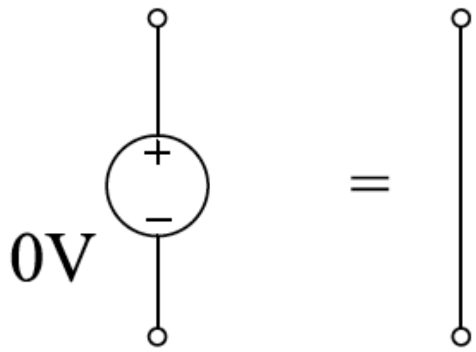


General Linearity

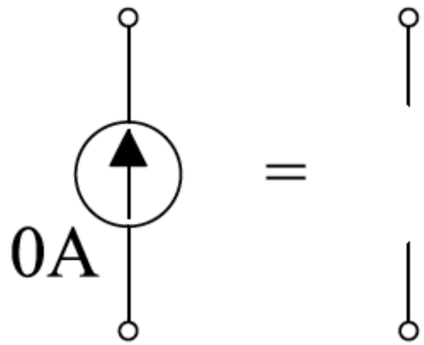


Zero-value sources

- A zero-valued voltage source has zero volts between its terminals for any current. It is equivalent to a *short-circuit* or piece of wire or resistor of 0 (or ∞S).



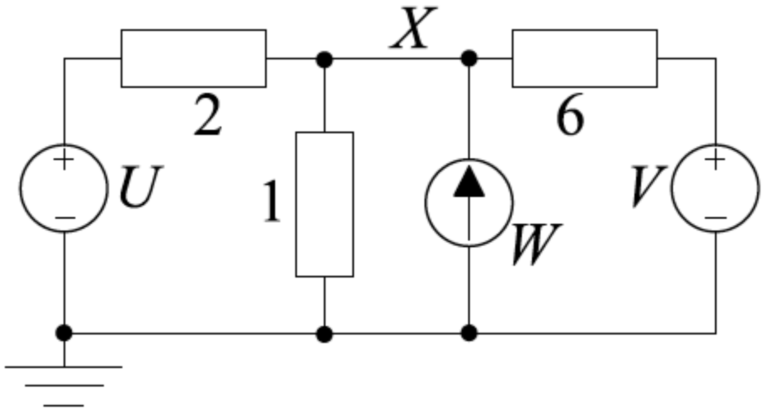
- A zero-valued current source has no current flowing between its terminals. It is equivalent to an *open-circuit* or a broken wire or a resistor of ∞ (or 0 S).

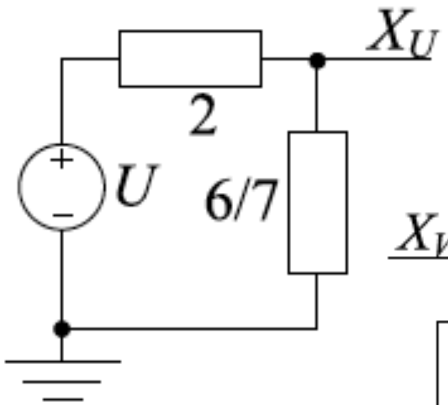


Superposition Calculation

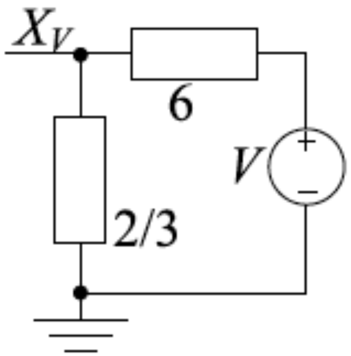
Superposition

Find the effect of each source on its own by setting all other sources to zero. Then add up the results.

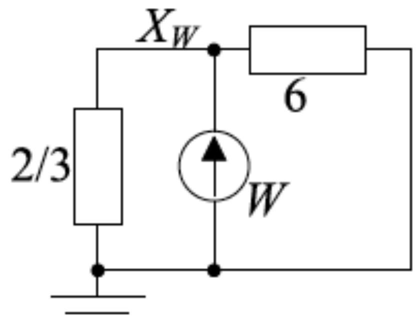




$$X_U = \frac{\frac{6}{7}}{2+\frac{6}{7}}U = \frac{6}{20}U = 0.3U$$



$$X_V = \frac{\frac{2}{3}}{6+\frac{2}{3}}V = \frac{2}{20}V = 0.1V$$



$$X_W = \frac{6}{6+\frac{2}{3}}W \times \frac{2}{3} = \frac{12}{20}W = 0.6W$$

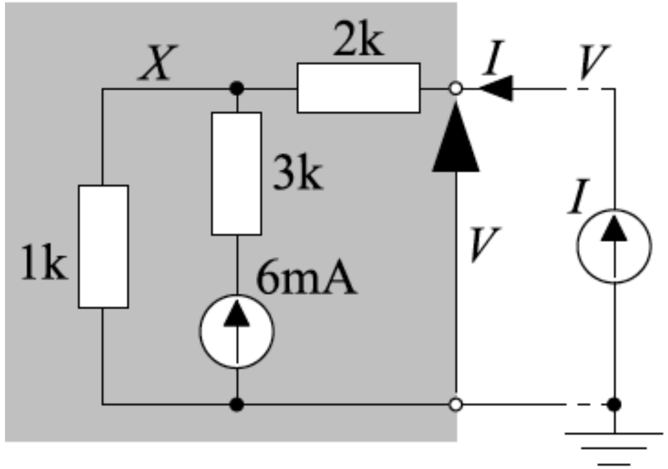
- Adding them up: $X = X_U + X_V + X_W = 0.3U + 0.1V + 0.6W$

Equivalent Networks

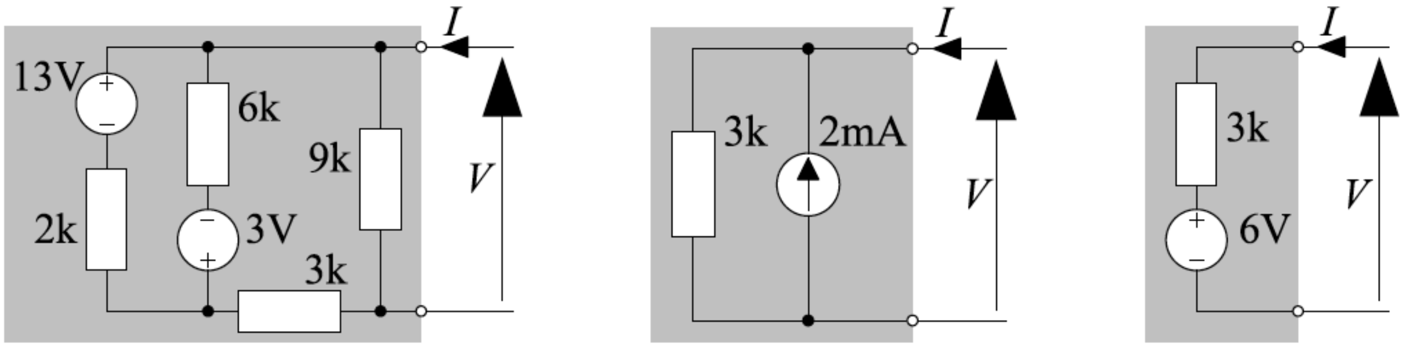
- From linearity theorem: $V = aI + b$.
- Use nodal analysis:

  KCL@X: $\frac{X}{1} - 6 + \frac{X-V}{2} = 0$

  KCL@V: $\frac{V-X}{2} - I = 0$



- Eliminating X gives: $V = 3I + 6$.
- There are infinitely many networks with the same values of a and b:



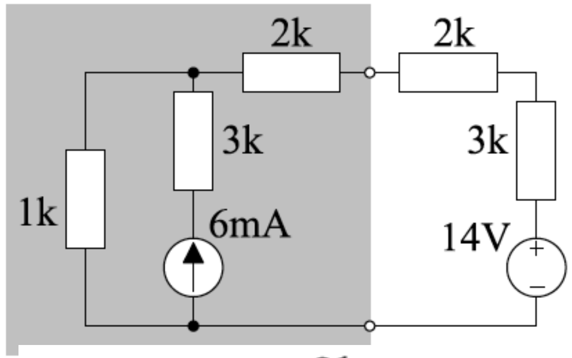
- These four shaded networks are *equivalent* because the relationship between V and I is *exactly* the same in each case.
- The last one is particularly simple and is called the *Thévenin* equivalent network.

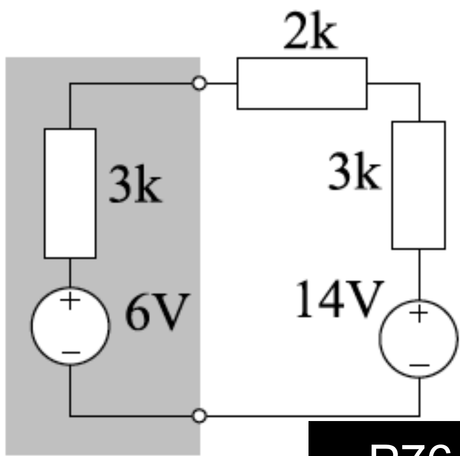
Thévenin Equivalent Circuit

Thévenin Theorem

Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in series with a fixed voltage source.

- We can replace the shaded part of the circuit with its Thévenin equivalent circuit.
- The voltages and currents in the unshaded part of the circuit will be identical in both circuits.
- The new components are called the *Thévenin equivalent resistance*, R_{Th}, and the *Thévenin equivalent voltage*, V_{Th}, of the original network.
- This is often a useful way to simplify a complicated circuit (provided that you do not want to know the voltages and currents inside the shaded part).





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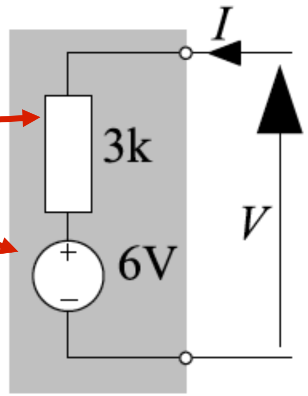
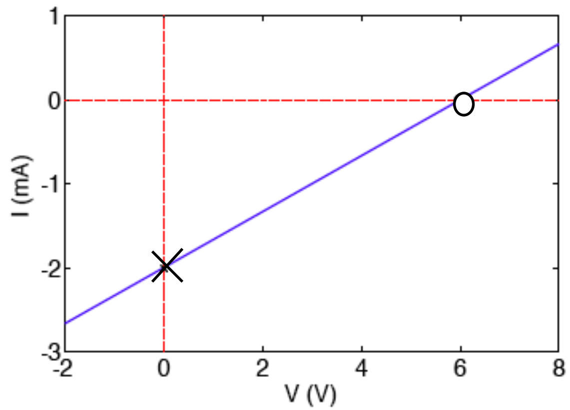
Thévenin Circuit Properties

- A Thévenin equivalent circuit has a straight line characteristic with the equation: $V = R_{Th} I + V_{Th}$

$$\Leftrightarrow I = \frac{1}{R_{Th}} V - \frac{V_{Th}}{R_{Th}}$$





- Three important quantities are:

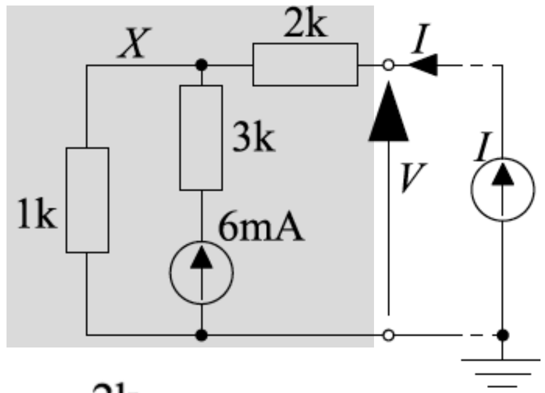
  Open Circuit Voltage: If $I = 0$ then $V_{OC} = V_{Th}$. (X-intercept: o)

  Short Circuit Current: If $V = 0$ then $I_{SC} = -\frac{V_{Th}}{R_{Th}}$ (Y-intercept: x)

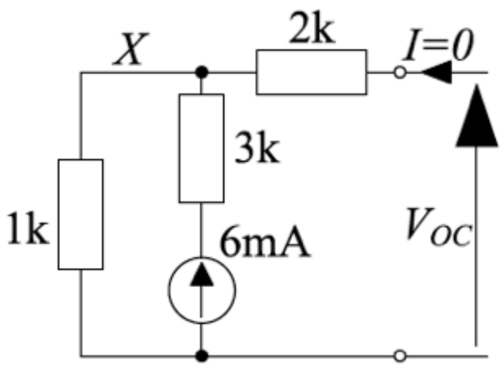
  Thévenin Resistance: The slope of the characteristic is. $\frac{dI}{dV} = \frac{1}{R_{Th}}$.

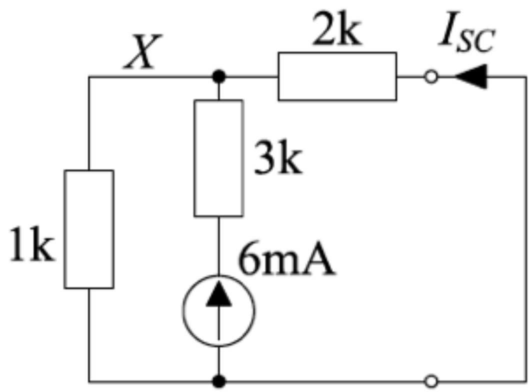
- If we know the value of any two of these three quantities, we can work out V_{Th} and R_{Th}.
- In any two-terminal circuit with the same characteristic, the three quantities will have the same values. So if we can determine two of them, we can work out the Thévenin equivalent.

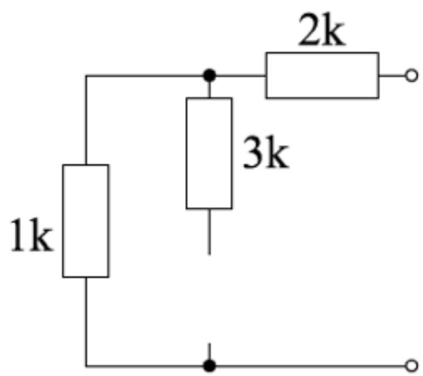
Determining Thévenin Values

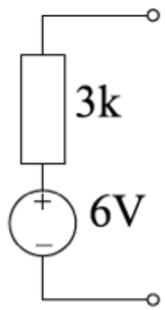


- We need any two of the following:
  - Open Circuit Voltage: $V_{OC} = V_{Th} = 6\,\mathrm{V}$
  - Short Circuit Current: $I_{SC} = -\frac{V_{Th}}{R_{Th}} = -2\,\mathrm{mA}$
  - Thévenin Resistance: $\mathrm{R}_{Th} = 2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}\Omega$







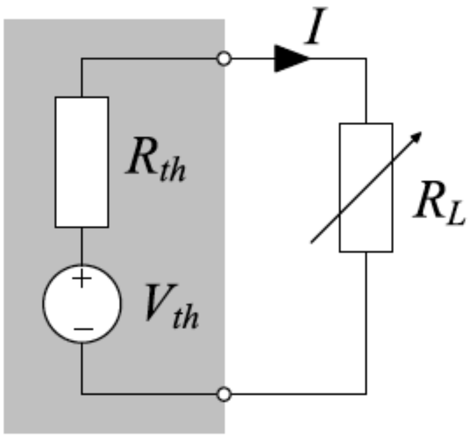


Thévenin Resistance:

- We set all the independent sources to zero (voltage sources → short circuit, current sources → open circuit). Then we find the equivalent resistance between the two terminals.
- The *3 k* resistor has no effect so $R_{Th} = 2\ k + 1\ k = 3\ k$.
- Any measurement gives the same result on the equivalent circuit.

Power Transfer

- Suppose we connect a variable resistor, RL, across a two-terminal network.
- From Thévenin's theorem, even a complicated network is equivalent to a voltage source and a resistor.



- We know $I = \frac{V_{Th}}{R_{Th}+R_L}$

  $\Rightarrow$ power in R_L is $P_L = I^2 R_L = \frac{V_{Th}^2 R_L}{(R_{th}+R_L)^2}$

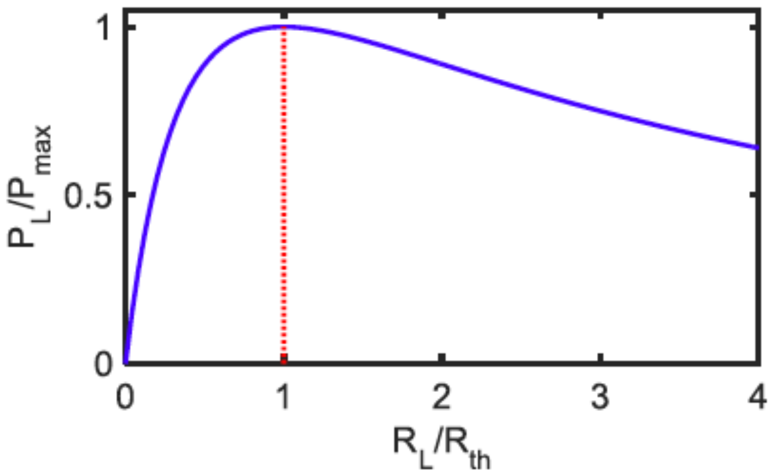
- To find the R_L that maximizes P_L:

$$0 = \frac{dP_L}{dR_L} = \frac{(R_{Th}+R_L)^2 V_{Th}^2 - 2V_{Th}^2 R_L (R_{Th}+R_L)}{(R_{Th}+R_L)^4}$$

$$= \frac{V_{Th}^2 (R_{Th}+R_L) - 2V_{Th}^2 R_L}{(R_{Th}+R_L)^3}$$

$$\Rightarrow V_{Th}^2 \left((R_{Th} + R_L) - 2R_L\right) = 0$$

$$\Rightarrow R_L = R_{Th} \;\Rightarrow\; P_{(max)} = \frac{V_{Th}^2}{4R_{Th}}$$



- For fixed R_{Th}, the maximum power transfer is when $R_L = R_{Th}$ ("matched load").

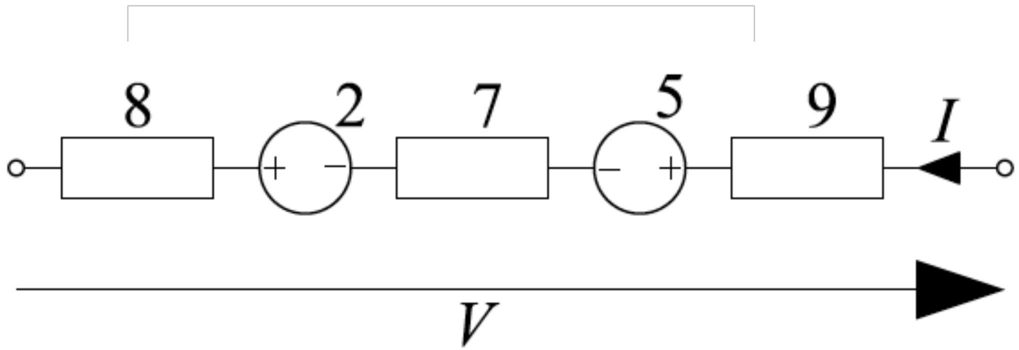
Series Rearrangement

- If we have any number of voltage sources and resistors in series we can calculate the total voltage across the chain as:

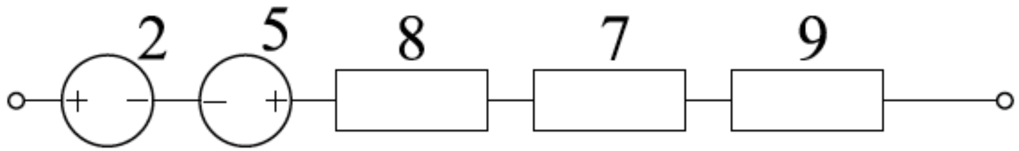
$$V = 8I - 2 + 7I + 5 + 9I = (-2+5) + (8+7+9)I$$

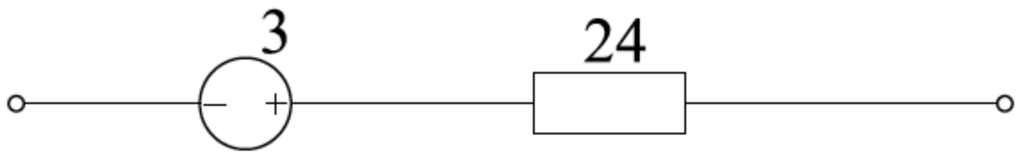
$$= 3 + 24I$$

- We can arbitrarily rearrange the order of the components without affecting V = 3 + 24I.



- If we move all the voltage sources together and all the resistors together we can merge them and then we get the Thévenin equivalent.





Summary

- Linearity Theorem: $X = \sum_i a_i U_i$ over all independent sources U_i
- Proportionality: multiplying all sources by k multiplies all voltages and currents by k and all powers by k^2.
- Superposition: sometimes simpler than nodal analysis, often more insight.
  - Zero-value voltage and current sources
- If all sources are fixed except for U_1 then all voltages and currents in the circuit have the form $aU_1 + b$.
- Power does not obey superposition.

- Thévenin Equivalent Circuits
  - How to determine V_{Th}, I_{NO} and R_{Th}
    - Method 1: Nodal analysis
    - Method 2: Find any two of $V_{OC} = V_{Th}$, I_{SC}
    - R_{Th} is the equivalent resistance with all sources set to zero
  - Ohm's law is satisfied: $V_{Th} = I_{NO} R_{Th}$
  - Load resistor for maximum power transfer = R_{Th}